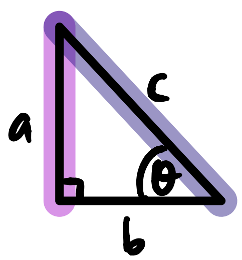
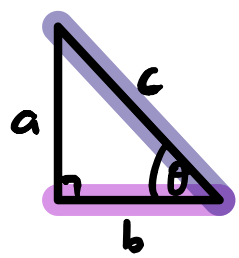
The Unit Circle

What we know about trig so far...

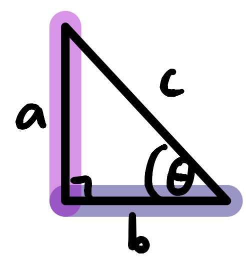
We can find trig ratios using right-angled triangles:



$\sin(\theta) = \frac{a}{c}$



$\cos(\theta) = \frac{b}{c}$



$\tan(\theta) = \frac{a}{b}$

What limitations do these definitions have?

Because we're working with right triangles, we must have

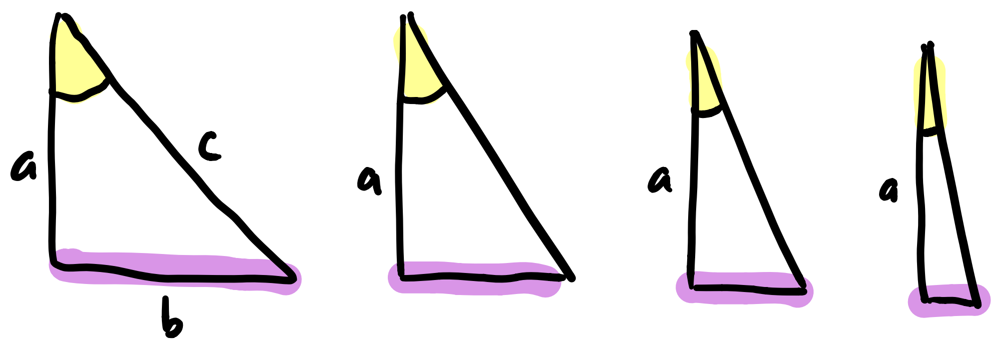
$$0 < \theta < 90^\circ$$

(angle sum of a triangle)

So we can only use trig when working with a relatively small number of angles.

One way to expand our definitions...

It's impossible to create a right triangle with a 0° angle, but let's consider what would happen as our angle became smaller and smaller...



What happens to sin of the indicated angle? Our opposite side is getting shorter and shorter (approaching 0 length) as our angle gets smaller and smaller (approaching 0°). Our hypotenuse is also changing in length - it's approaching the length of the third side (which is the only side that doesn't change in length)

So $\sin(\theta) \to \frac{0}{a} = 0$

and $\cos(\theta) \rightarrow \frac{a}{a} = 1$

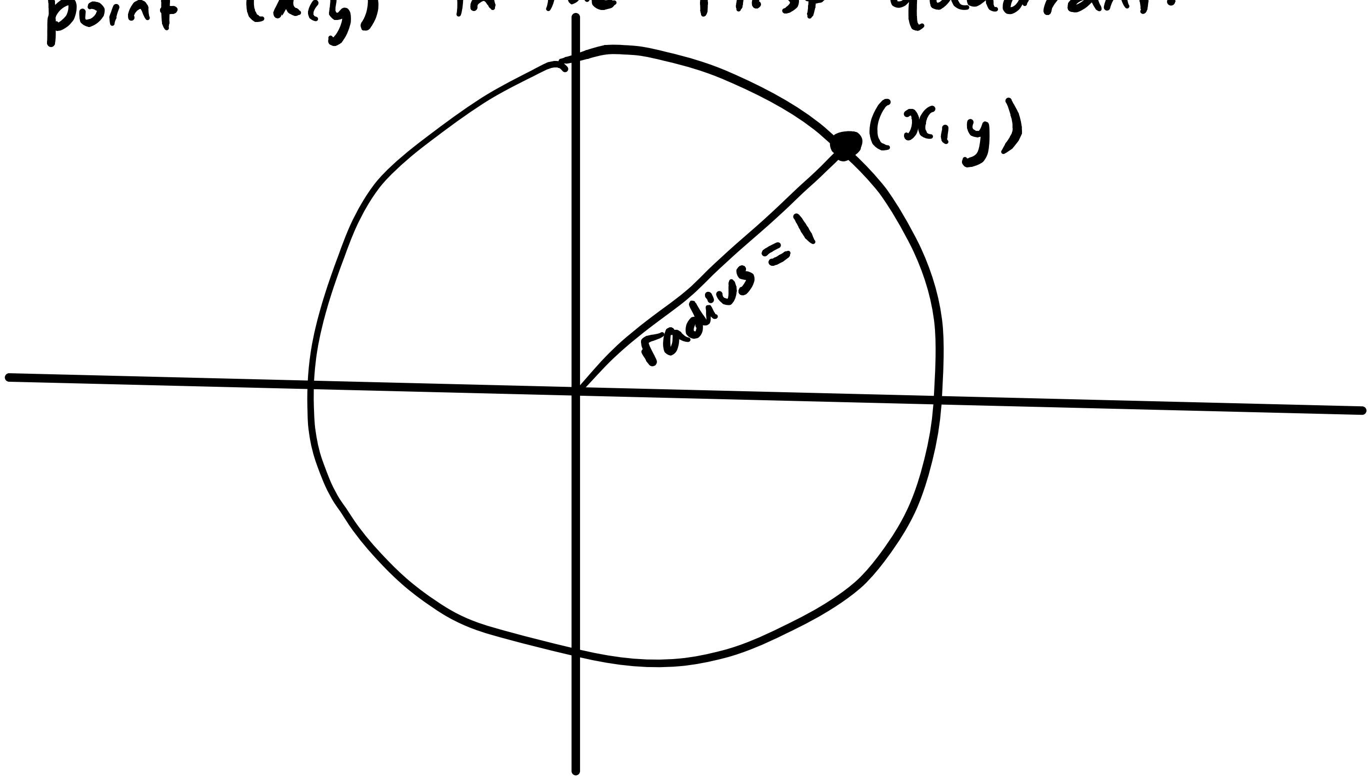
It would make sense to then say

$$\sin(0) = 0, \qquad \cos(0) = 1$$

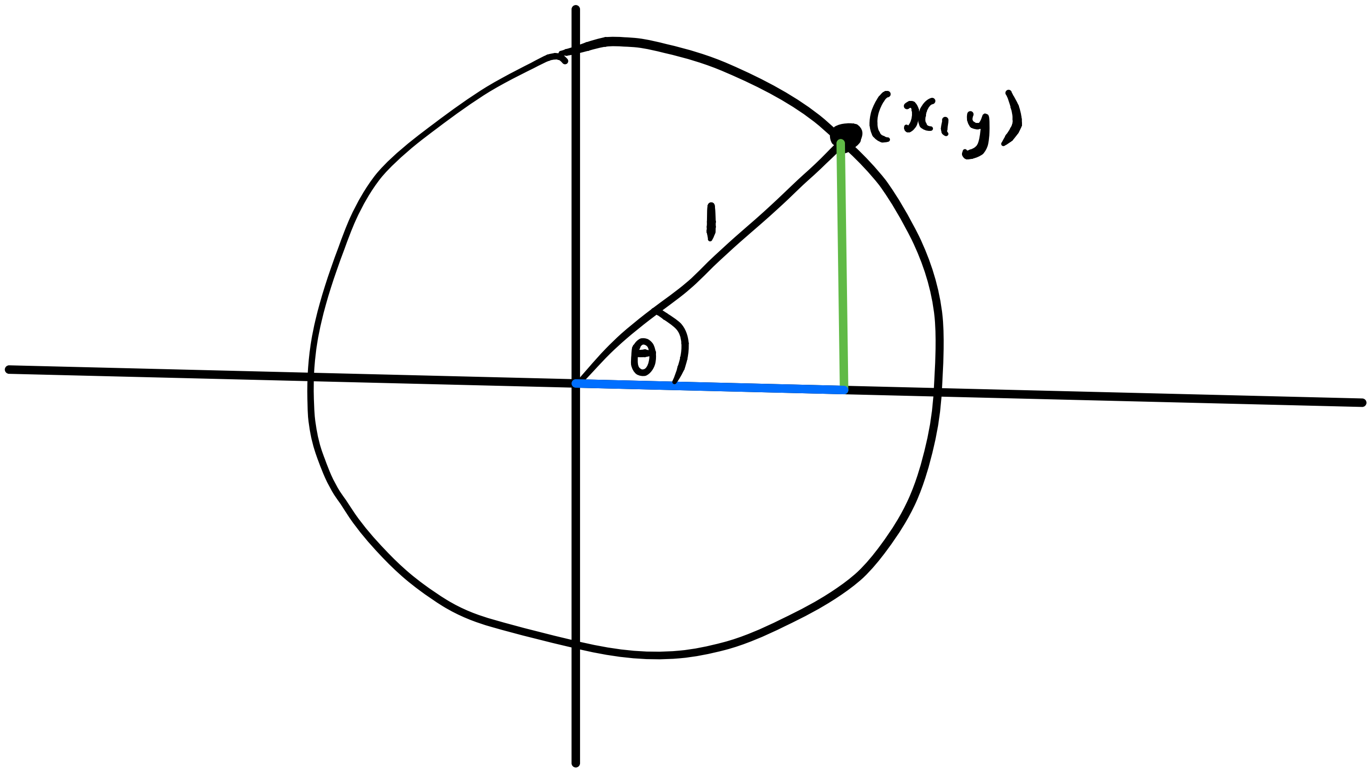
Unfortunately, this method only gave us one extra angle to use.

How do we extend our definitions further? Can we apply trig functions to angles greater than 90°? How about negative numbers?

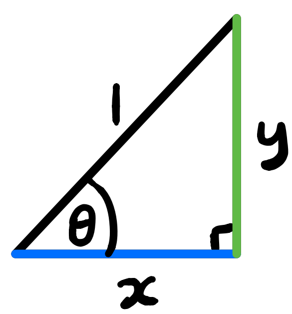
Here's where the unit circle comes in. Let's graph it : $x^2+y^2=1$ and consider a point (x,y) in the first quadrant.



We can create a triangle:



Using the coordinates of our point, we can find the blue and green distances:



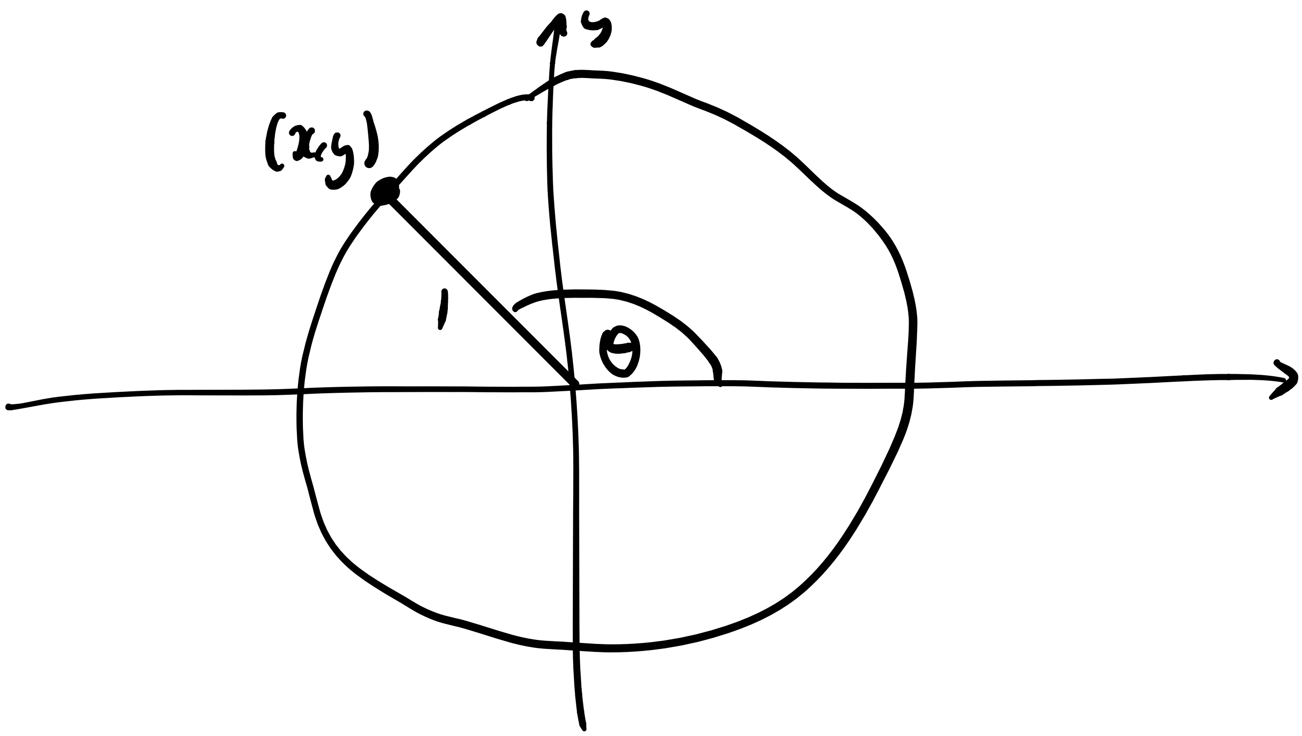
Because (x, y) is in the first quadrant, θ (the angle between the radius and x-axis) is between $0°$ and $90°$.

So we can use our trig ratios to find:

$$\cos(\theta) = \frac{x}{1} = x$$

$$\sin(\theta) = \frac{y}{1} = y$$

Using this, we will <u>define</u> trig functions for any $\theta \in [0; 360^\circ]$ as follows:



① Select the point (x,y) on the unit circle such that the angle between the radius and the positive x-axis is θ.

② Set $\cos(\theta) := x$, $\sin(\theta) := y$

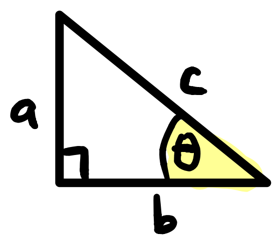
This is consistent with our definitions of trig ratios for $\theta \in (0; 90^\circ)$, and allows us to apply trig functions to any

$\theta \in [0, 360^\circ]$.

Where does the 360 come from? The angle of revolution at a point (the origin).

Pythagoras' Theorem and Circles

Remember that for a right-angled triangle



we have

$$a^2 + b^2 = c^2$$

and also $\cos^2(\theta) + \sin^2(\theta) = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2$

$$= \frac{b^2 + a^2}{c^2}$$

$$= \frac{c^2}{c^2}$$

$$= 1$$

This result holds for any $\theta \in (0, 90^\circ)$ since we used a right-angled triangle. For $\theta \in [0, 360^\circ]$ we can use the unit circle:

$$\cos^2(\theta) + \sin^2(\theta) = x^2 + y^2$$

since $x = \cos(\theta)$
$y = \sin(\theta)$

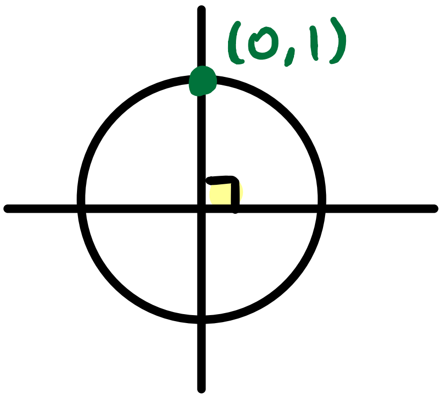
$$= 1$$

using the equation of the circle.

This is called the Pythagorean Identity.

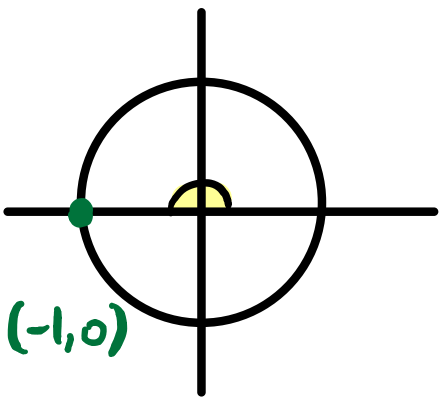
How can we use the unit circle?

We can easily find some trig ratios just by reading off coordinates:



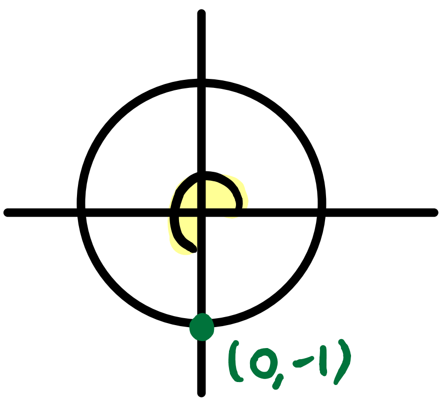
$\cos(90^\circ) = 0$

$\sin(90^\circ) = 1$



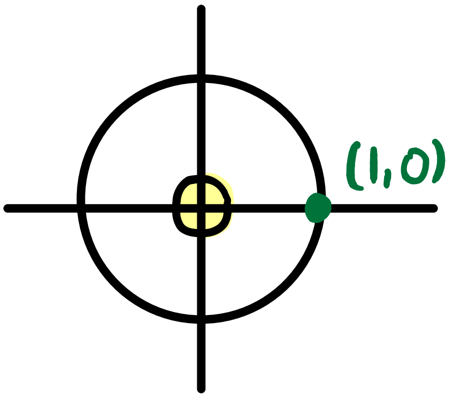
$\cos(180^\circ) = -1$

$\sin(180^\circ) = 0$



$\cos(270^\circ) = 0$

$\sin(270^\circ) = -1$



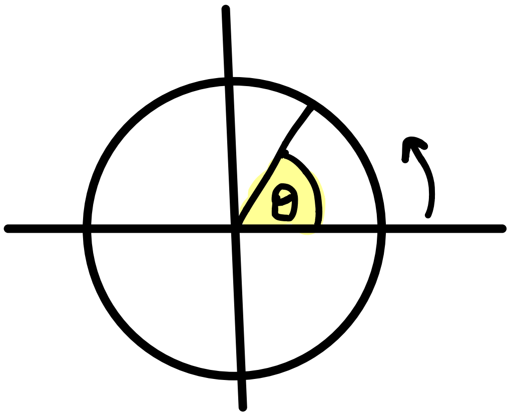
$\cos(360^\circ) = 1$
$\sin(360^\circ) = 0$

Notice that we have periodicity occurring here - once the angle gets back around to the x-axis, we get back to the first quadrant and everything "starts over". For any point on the unit circle, we have no idea how many times our angle has "gone round" the origin, so we can conclude that

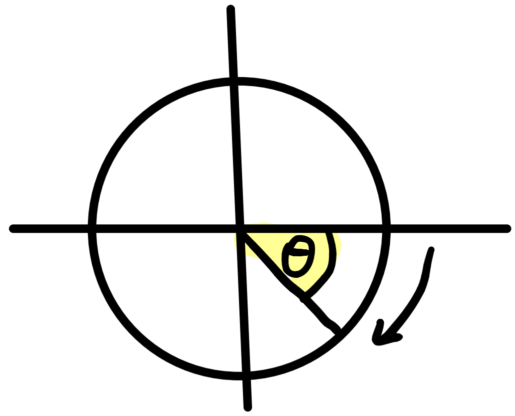
$$\cos(x) = \cos(x + 360^\circ)$$
$$\sin(x) = \sin(x + 360^\circ)$$

How about negative numbers?

We created an angle of $\theta > 0$ by moving anti-clockwise from the x-axis. To create a negative angle, we will move clockwise (the opposite direction):



Positive angle

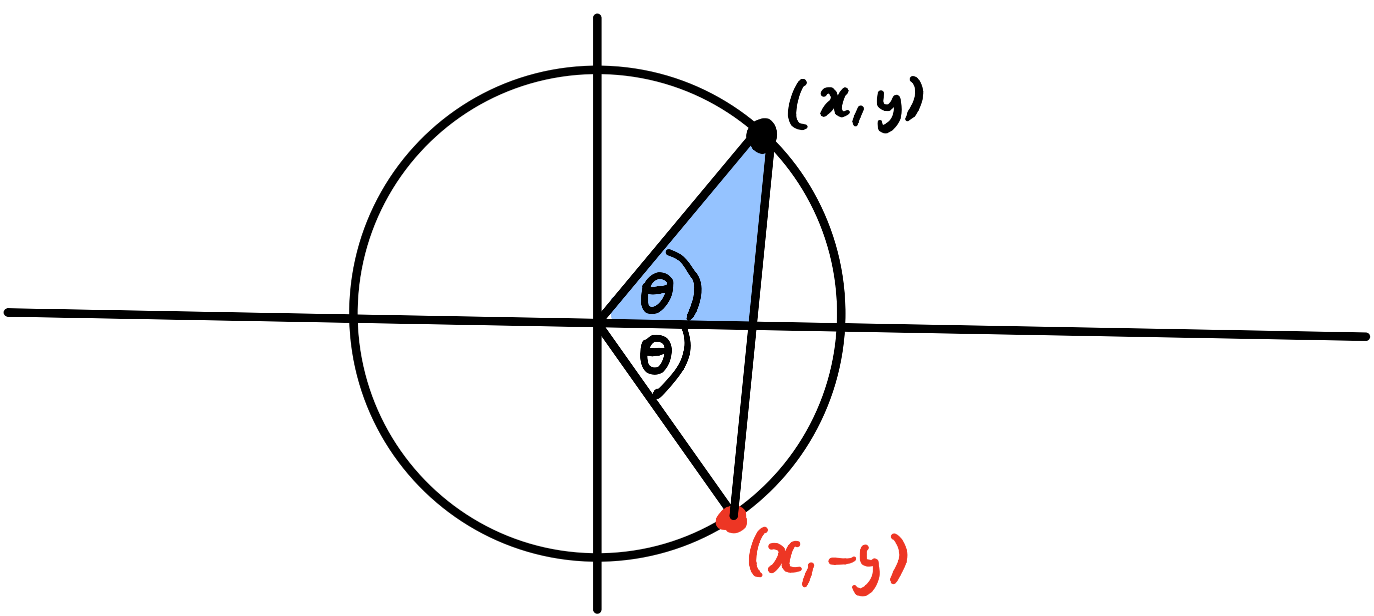


Negative angle

Keeping in mind that the positive x-axis represents both 0° and 360°, we have

$$\cos(-x) = \cos(360^\circ - x)$$
$$\sin(-x) = \sin(360^\circ - x)$$

But also, by looking at the diagram we can see that:

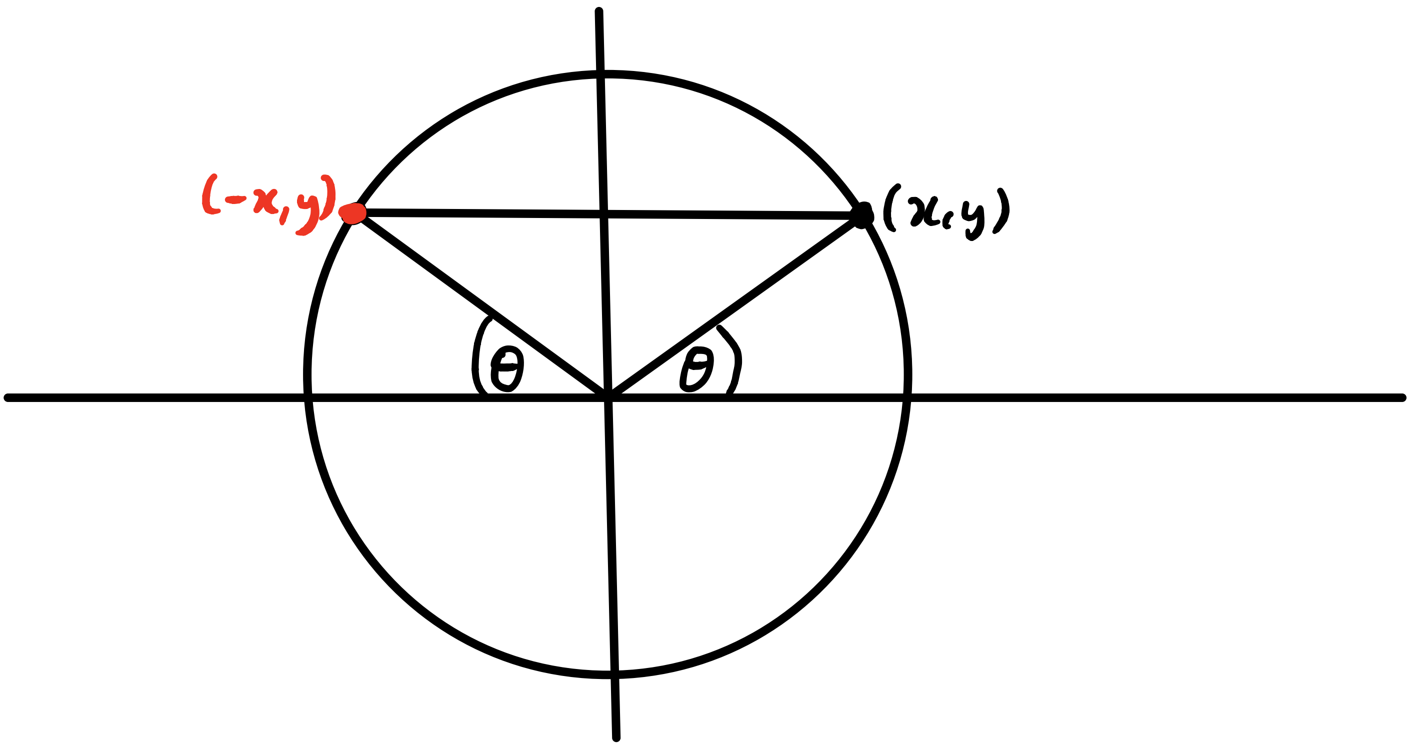


To Find the coordinates of the red point, we Flipped the blue triangle across the x-axis. So the x-coordinate stayed the same, and the y-coordinate became negative.

We can then say that

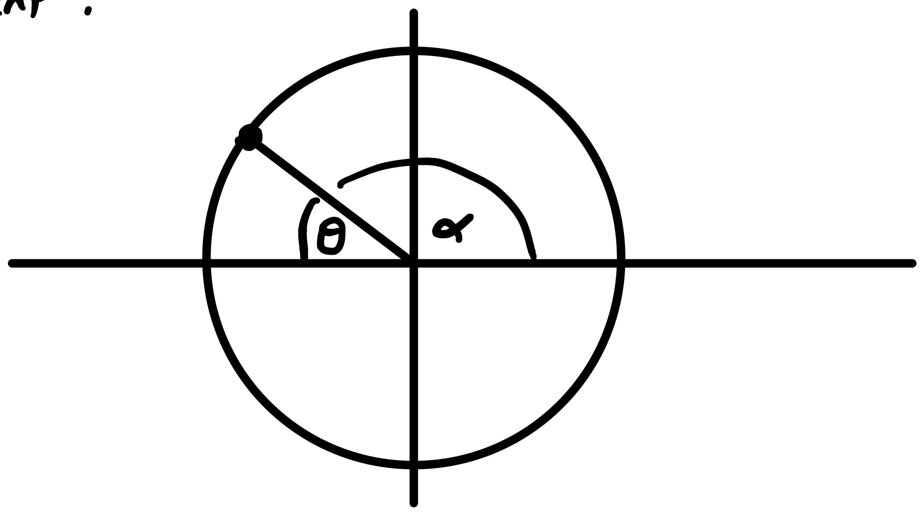
$$\boxed{\begin{aligned} \cos(-x) &= \cos(x) \\ \sin(-x) &= -\sin(x) \end{aligned}}$$

More Trig Identities!



This time we Flip the point across the y-axis, not the x-axis. So the x-coordinate becomes negative and the y-coordinate stays the same.

The question is, what does the red point represent?



We want to find α - the angle between the radius drawn in, and the positive x-axis. But

$$\theta + \alpha = 180^\circ$$

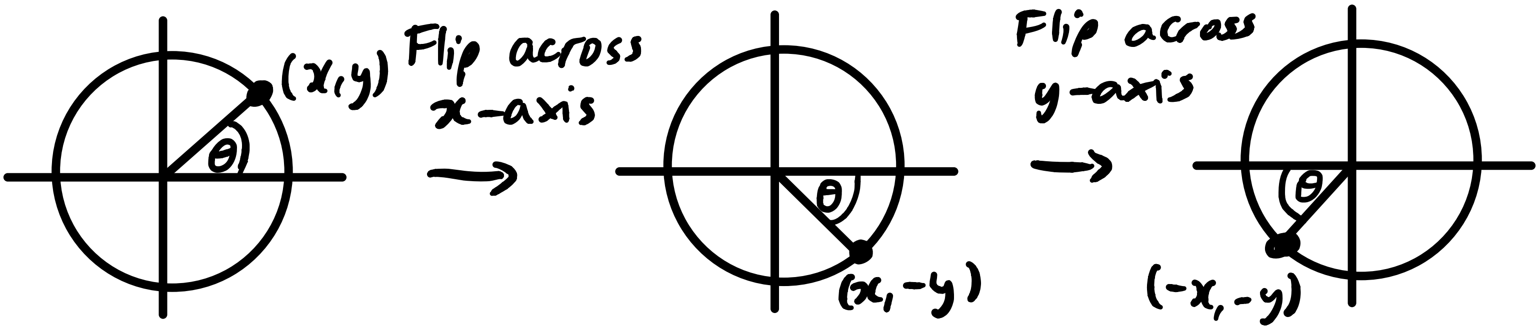
$$\Rightarrow \alpha = 180^\circ - \theta$$

So

$$\cos(180^\circ - \theta) = -\cos(\theta)$$

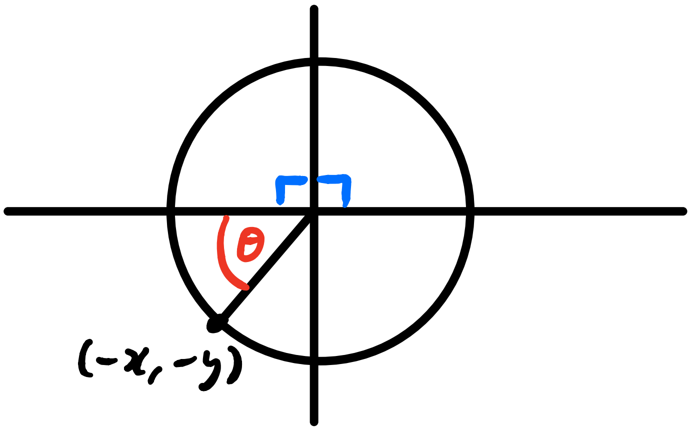
$$\sin(180^\circ - \theta) = \sin(\theta)$$

We can also flip our point across both the x- and y- axes:



The angle we're now working with is

$180° + \theta$



So we have

$$\cos(\theta + 180°) = -\cos(\theta)$$

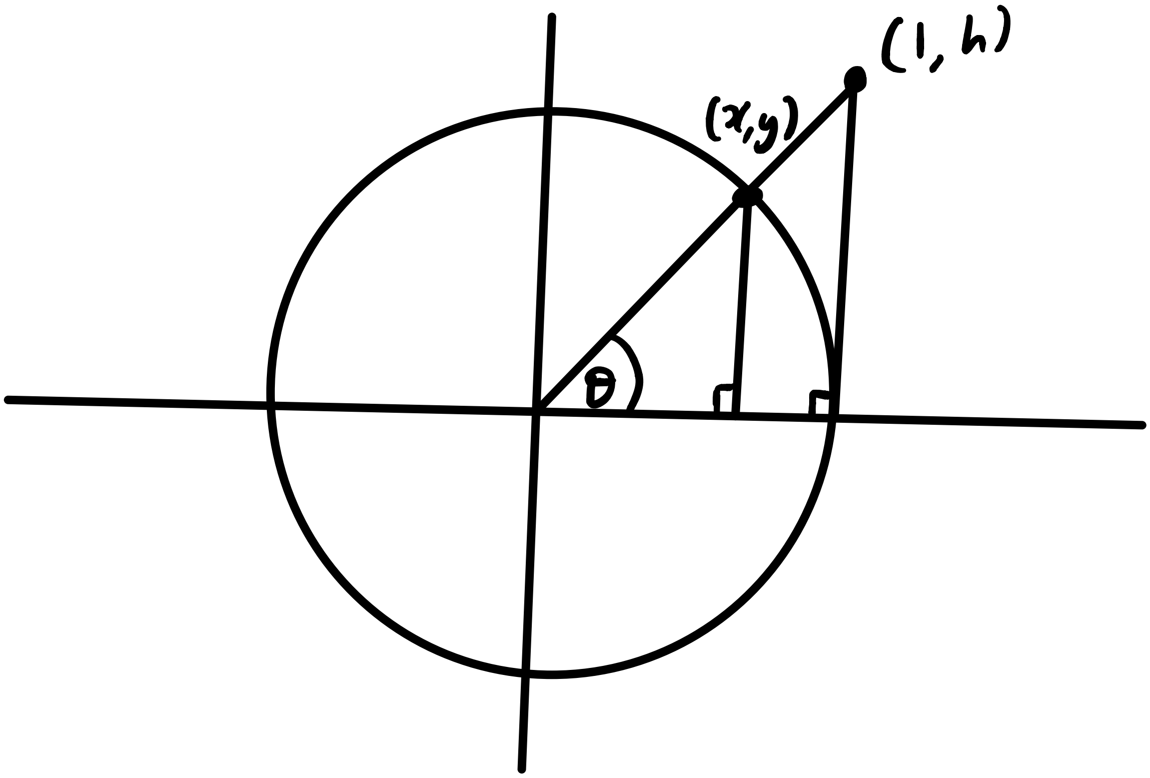
$$\sin(\theta + 180°) = -\sin(\theta)$$

How about tan?

As with trig ratios in right triangles,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

We also see where the name "tan" comes from - it's short for "tangent":



① Draw the tangent to the circle at $(1,0)$ (the line $x = 1$)

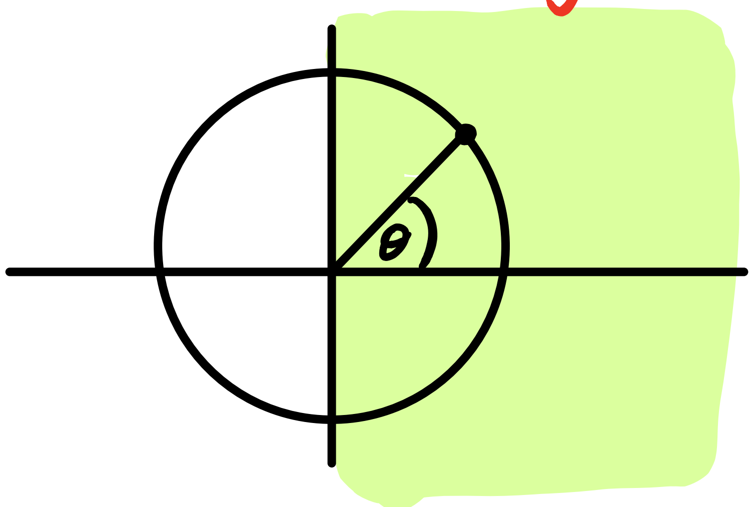
② Extend the line from $(0,0)$ to (x,y) until it meets the tangent

③ Find h :

$$\tan(\theta) = \frac{y}{x} = \frac{h}{1}$$

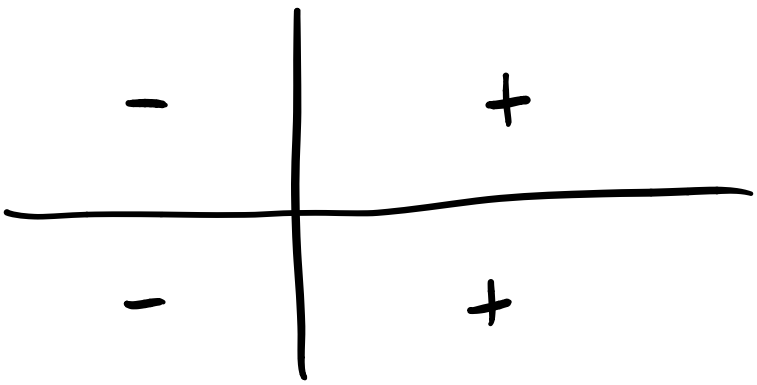
So $\tan(\theta)$ is the length of the tangent segment!

Positive and Negative

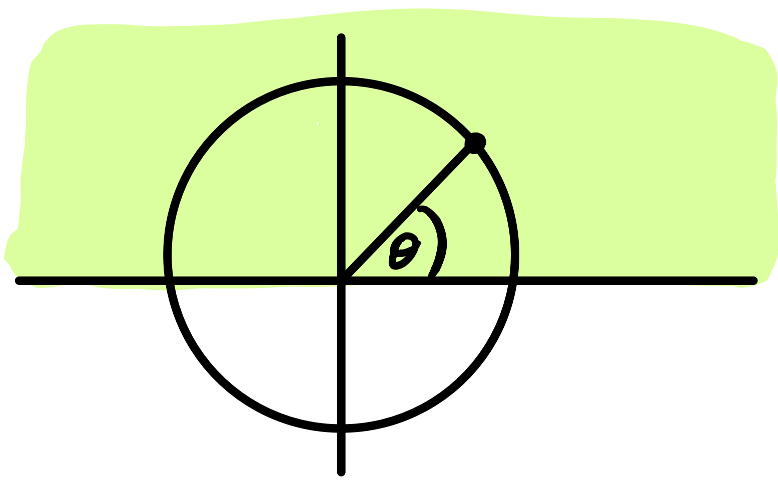


When $x > 0$,

$\cos(\theta) > 0$

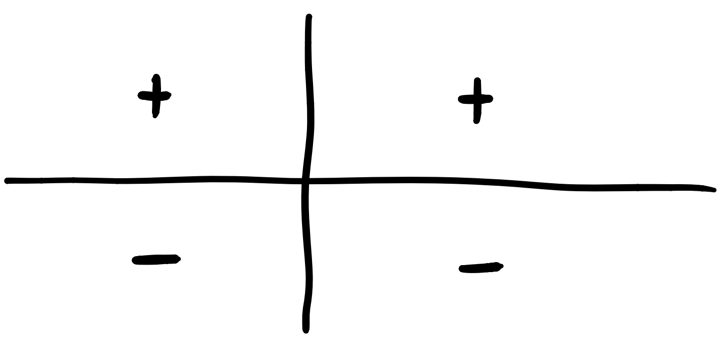


So $\cos(\theta)$ is positive in the 1st and 4th quadrants



When $y > 0$,

$\sin(\theta) > 0$



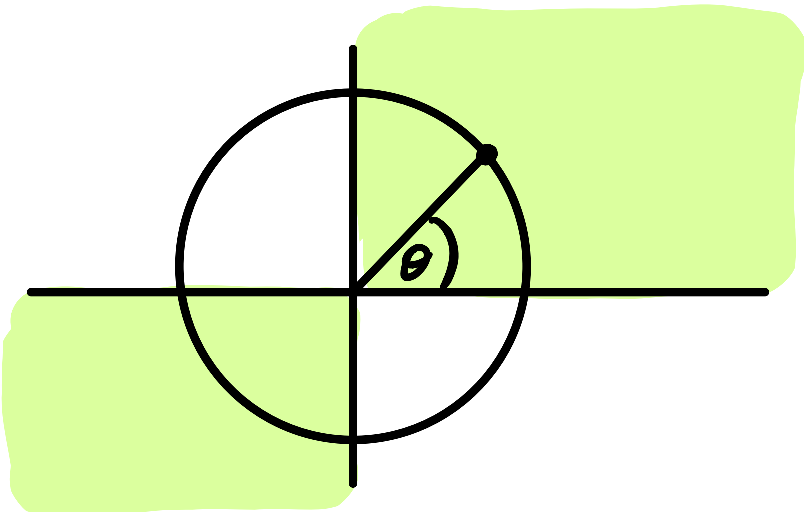
So $\sin(\theta)$ is positive in the 1st and 2nd quadrants

Now, $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ so $\tan(\theta)$ is positive when either

$$\cos(\theta),\ \sin(\theta) > 0 \quad \text{(both)}$$

or

$$\cos(\theta),\ \sin(\theta) < 0 \quad \text{(both)}$$



$\tan(\theta)$ is positive in the 1st and 3rd quadrants

Also,

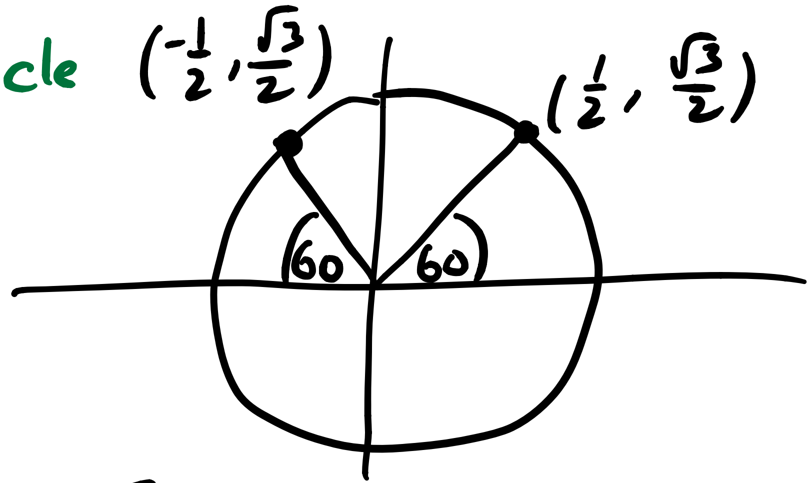
$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)}$$

$$= \frac{-\sin(\theta)}{\cos(\theta)}$$

$$= -\left[\frac{\sin(\theta)}{\cos(\theta)}\right]$$

$$= -\tan(\theta)$$

Using the Unit Circle



$\sin(120^\circ) = ?$

$120 = 180 - 60$

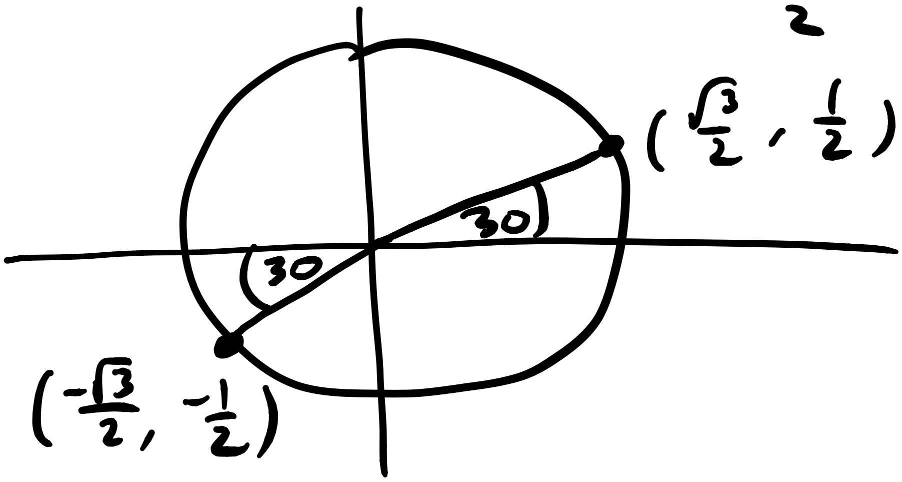
so

$\sin(120^\circ) = \sin(60^\circ) = \frac{\sqrt{3}}{2}$

$\cos(120^\circ) = ?$

Using the same diagram, $\cos(120^\circ) = -\cos(60^\circ)$

$= -\frac{1}{2}$



$\tan(210^\circ) = ?$

We have

$210 = 180 + 30$

So

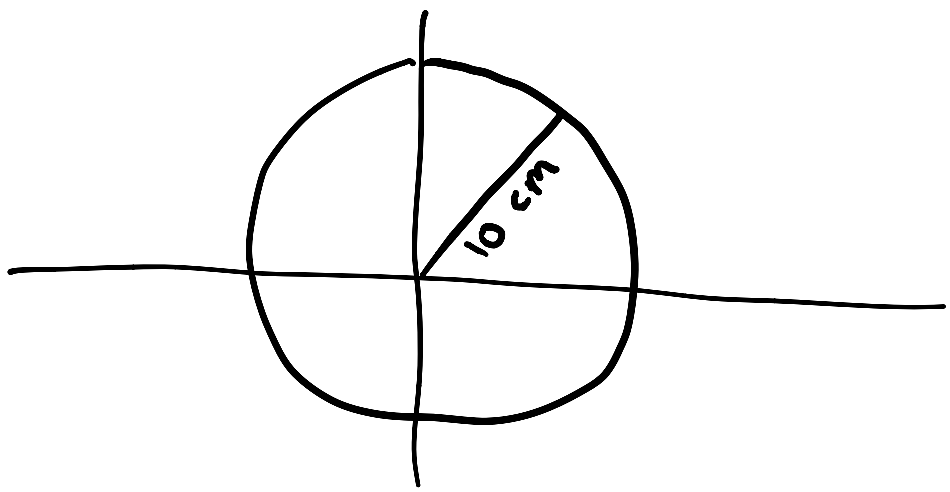
$\tan(210^\circ) = \tan(30^\circ) = \frac{1}{\sqrt{3}}$

<u>OR</u> $= \dfrac{\sin(210^\circ)}{\cos(210^\circ)}$ (from the diagram)

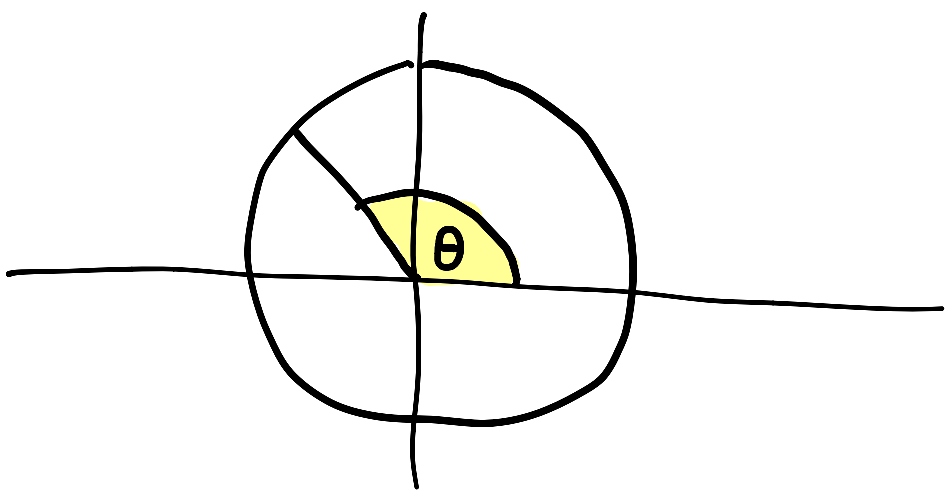
$= \dfrac{\left(-\frac{1}{2}\right)}{\left(-\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}}$ as expected

Paper and Pencil Activity!

1. Draw a circle using a compass. You can use any radius, but I would suggest 10cm so that your diagram is big enough to measure accurately and see easily. 10 is also an easy number to calculate with.

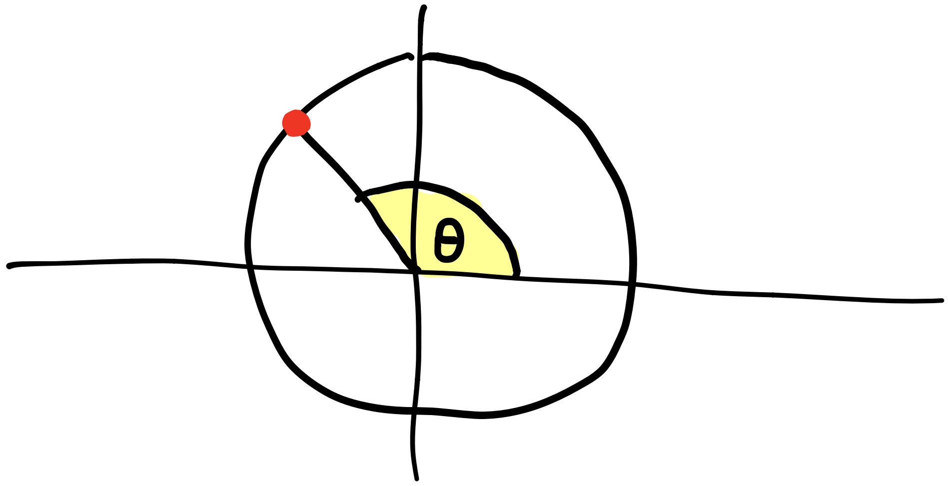


2. Choose an angle and measure it out with your protractor.



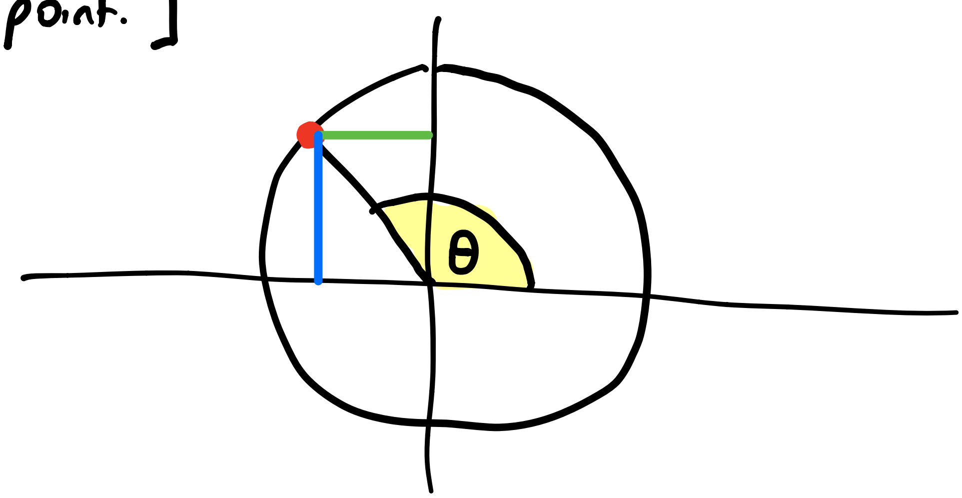
3. Mark the corresponding point on the

circle.



4. Find $\cos(\theta)$ and $\sin(\theta)$ by measuring the corresponding distances with a ruler then dividing by 10 (or whatever your choice of radius was). Remember the + or − depending on the quadrant you're in!

[We are finding the x and y coordinates of red point.]



5. Check your findings using a calculator. How close were you??