Mathematical Methods 2019 v1.2

IA2 sample assessment instrument

November 2019

Examination (15%)

This sample has been compiled by the QCAA to assist and support teachers in planning and developing assessment instruments for individual school settings.

Assessment objectives

This assessment instrument is used to determine student achievement in the following objectives:

1. select, recall and use facts, rules, definitions and procedures drawn from all Unit 3 topics
2. comprehend mathematical concepts and techniques drawn from all Unit 3 topics
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from all Unit 3 topics.

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Subject	Mathematical Methods	Instrument no.	IA2
Technique	Examination		
Unit	3: Further calculus		
Topic	1: The logarithmic function 2 2: Further differentiation and applications 2 3: Integrals		

Conditions			
Response type	Short response		
Time	Paper 1: 60 minutes Paper 2: 60 minutes	Perusal	5 minutes (Paper 2)
Other	• QCAA formula sheet • Non-CAS graphics calculator		

Instructions

- Show all working in the space provided.
- Use a black or blue pen.
- Use of a non-CAS graphics calculator is permitted in the technology-active paper only.

Criterion	Marks allocated	Result
Foundational knowledge and problem-solving Assessment objectives 1, 2, 3, 4, 5 and 6	15	

Paper 1 (technology-free) — total marks: 69

Question 1 (11 marks)

Solve each equation below:

a. $x = \log_6 36$ $6^2 = 36 \Rightarrow x = 2$

b. $\log_3(4x - 7) = 2$ $\Rightarrow 4x - 7 = 3^2 = 9$

$\Rightarrow 4x = 16 \Rightarrow x = 4$

c. $(e^x - 2)(e^x - 3) = 0$ $e^x = 2$ or $e^x = 3$

$\Rightarrow x = \ln(2)$ or $x = \ln(3)$

d. $\ln x + \ln(2-x) = 0$ $\Rightarrow \ln[x(2-x)] = 0$

$\Rightarrow x(2-x) = 1 \Rightarrow -x^2 + 2x - 1 = 0$

$\Rightarrow x^2 - 2x + 1 = 0 \Rightarrow (x-1)^2 = 0$

$\Rightarrow x = 1$

Question 2 (16 marks)

Determine the derivative of the following functions:

a. $f(x) = e^x + \sin(2x)$

$f'(x) = e^x + 2\cos(2x)$

b. $f(x) = e^{\sin(x)}$

$f'(x) = e^{\sin(x)} \cdot \cos(x)$

c. $f(x) = \cos^3(x)$

$f'(x) = 3\cos^2(x)[-\sin(x)]$

d. $f(x) = x + x\ln(x)$ (give solution in simplest form)

$f'(x) = 1 + x \cdot \frac{1}{x} + \ln(x) = 1 + 1 + \ln(x)$

$= 2 + \ln(x)$

e. $f(x) = \frac{2x+3}{x^2+3x}$ (give solution in simplest form) $u := 2x+3, \quad v := x^2+3x$

$u' = 2, \quad v' = 2x+3$

$f'(x) = \frac{u'v - uv'}{v^2}$

$= \frac{2(x^2+3x) - (2x+3)^2}{(x^2+3x)^2}$

$= \frac{2x^2 + 6x - 4x^2 - 12x - 9}{(x^2+3x)^2}$

$= \frac{-2x^2 - 6x - 9}{(x^2+3x)^2}$

Question 3 (11 marks)

Determine the exact value for each of the following definite integrals (give the solutions in simplest form):

a. $\int_1^3 4x^2 dx$ $= \left[4\frac{x^3}{3}\right]_1^3 = \frac{4 \cdot 3^3}{3} - \frac{4 \cdot 1^3}{3}$

$= 4 \cdot 9 - \frac{4}{3} = 36 - \frac{4}{3} = \frac{104}{3}$

b. $\int_0^2 6e^{2t} + t\, dt$ $= \left[6e^{2t}\frac{1}{2} + t^2 \cdot \frac{1}{2}\right]_0^2$

$= 3e^4 - 3 + \frac{4}{2}$

$= 3e^4 - 1$

c. $\int_0^2 \frac{4x}{x^2+4} dx$ by considering the derivative of $f(x) = \ln(x^2 + 4)$

We have $f'(x) = \frac{2x}{x^2+4}$

So $\int_0^2 \frac{4x}{x^2+4} dx = 2\int_0^2 \frac{2x}{x^2+4} dx$

$= 2\left[\ln(x^2+4)\right]_0^2 = 2\left[\ln(8) - \ln(4)\right] = 2\ln\left(\frac{8}{4}\right)$

$= 2\ln(2)$

$= \ln(4)$

Question 4 (7 marks)

The model for the average height of a eucalyptus tree is given below:

$H = 6 + 6\log_3 t$, $t \geq 1$ where t = time in years, and H is the height in metres.

Determine:

a. the height of the tree after one year (give solution in simplest form)

$H(1) = 6 + 6\log_3(1) = 6 + 0 = 6$ m

b. the time it takes for the tree to reach a height of 18 metres.

$18 = 6 + 6\log_3(t) \Rightarrow t = 3^2 = 9$ yrs

$\Rightarrow 12 = 6\log_3(t)$

$\Rightarrow 2 = \log_3(t)$

Question 5 (6 marks)

A particle moves along the x-axis with position at time t given by $x(t) = e^t \sin(t)$ for $0 \le t \le 2\pi$. Calculate each time t for which the particle is at rest.

Particle at rest $\Rightarrow$ velocity $= 0 \Rightarrow x'(t) = 0$

$x'(t) = e^t \sin(t) + e^t \cos(t)$ so $x'(t) = 0$

$\Rightarrow e^t[\sin(t) + \cos(t)] = 0 \Rightarrow \sin(t) + \cos(t) = 0$

Since $e^t > 0 \;\forall t$.

See page below For soln.

Question 6 (6 marks)

Determine the coordinates of the maximum point of the function $f(x) = \frac{\ln(2x)}{x}$, $x > 0$.

$u = \ln(2x)$ $v = x$

$u' = \frac{2}{2x} = \frac{1}{x}$ $v' = 1$

$$f'(x) = \frac{u'v - uv'}{v^2}$$

$$= \frac{\frac{1}{x}x - \ln(2x)}{x^2} = \frac{1 - \ln(2x)}{x^2}$$

$f'(x) = 0 \iff 1 - \ln(2x) = 0$

$\iff \ln(2x) = 1$

$\iff 2x = e$

$\iff x = e/2$

See second page below

How to solve $\sin(t) + \cos(t) = 0$?

We set $LHS = R\sin(t - \theta)$

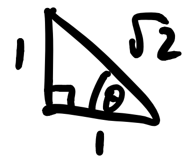
$$= R\sin(t)\cos(\theta) - R\sin(\theta)\cos(t)$$

Equating coeffs :

$$R\cos(\theta) = 1 \qquad ①$$

$$-R\sin(\theta) = 1 \Rightarrow R\sin(\theta) = -1 \qquad ②$$

② ÷ ① : $\tan(\theta) = -1$



$\Rightarrow \theta = -\frac{\pi}{4}$ [since $\tan(\pi/4) = 1$ and tan is an odd function]

Sub into ① : $R\cos\left(-\frac{\pi}{4}\right) = 1$

$\Rightarrow R\cos\left(\frac{\pi}{4}\right) = 1$ [since cos is an odd fn]

$\Rightarrow R \cdot \frac{1}{\sqrt{2}} = 1$

$\Rightarrow R = \sqrt{2}$

So $\sqrt{2}\sin\left(t + \frac{\pi}{4}\right) = 0$

$\Rightarrow \sin\left(t + \frac{\pi}{4}\right) = 0$

$\Rightarrow t + \frac{\pi}{4} = n\pi \qquad (n \in \mathbb{Z})$

$$= \cdots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$$

$$\Rightarrow t = \ldots, \frac{-5\pi}{4}, \frac{-\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$$

Now, we know $0 \leq t \leq 2\pi$ so the only valid solutions are

$$t = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Checking For max:

When $x = e/2$, $f(x) = \dfrac{\ln(2 \cdot e/2)}{e/2}$

$$= \frac{\ln(e)}{e/2}$$

$$= \frac{1}{e/2}$$

$$= \frac{2}{e} \simeq 0.736..$$

When $x = \frac{e}{2} - 0.1$, $f(x) = \dfrac{\ln(e - 0.2)}{\frac{e}{2} - 0.1}$

$$\simeq 0.733.. < 0.736$$

When $x = \frac{e}{2} + 0.1$, $f(x) = \ln(e + 0.2) \simeq 0.734$

$\overline{e/2 + 0.1} < 0.736$

$\therefore \left(\frac{e}{2}, \frac{2}{e}\right)$ is a maximum

Question 7 (5 marks)

The spread of a flu in a certain school is modelled by the equation $P(t) = \frac{100}{1+e^{b-t}}$ where $P(t)$ is the total number of students infected after t days.

Given that the population at day 3 is 50, determine the rate the flu is spreading at this time.

$P(t) = 100(1+e^{b-t})^{-1}$

$\Rightarrow P'(t) = 100(-1)(1+e^{b-t})^{-2}(-e^{b-t})$

$P(3) = 50 \Rightarrow 50 = 100(1+e^{b-3})^{-1}$

$\Rightarrow \frac{1}{2} = (1+e^{b-3})^{-1} \Rightarrow 2 = 1+e^{b-3}$

$\Rightarrow 1 = e^{b-3} \Rightarrow b-3=0 \Rightarrow b=3$

So $P'(3) = -100(1+e^{0})^{-2}(-e^{0})$

$= -100 \times \frac{1}{4} \times -1 = 25$ students/day

Question 8 (7 marks)

A particle is moving on a straight line. Velocity varies with displacement according to the rule $v = \sqrt{4+4s}$, where v is velocity in $ms^{-1}$and s is the displacement in metres.

Calculate the displacement at $t = 2$ if the initial displacement is 0.

$\frac{ds}{dt} = (4+4s)^{1/2} = 2(1+s)^{1/2}$

let $x = 1+s$
$\Rightarrow dx = ds$

$\Rightarrow \frac{1}{2}\int \frac{1}{(1+s)^{1/2}}\, ds = \int 1\, dt$

$\Rightarrow \frac{1}{2}\int x^{-1/2}\, dx = t + C$

$\Rightarrow \frac{1}{2}\frac{x^{1/2}}{1/2} = t + C$

$\Rightarrow x^{1/2} = t + c \Rightarrow \sqrt{1+s} = t + c$

When $t=0$, $s=0 \Rightarrow \sqrt{1+0} = 0 + c \Rightarrow c = 1$

$\therefore \sqrt{1+s} = t+1 \Rightarrow 1+s = (t+1)^2$

$\Rightarrow s = (t+1)^2 - 1 = t^2 + 2t$

When $t = 2$, $s = 4+4 = 8m$

Paper 2 (technology-active) — total marks: 56

Question 1 (11 marks)

Sketch $f(x) = \frac{\ln(x)}{x}$, indicating important features, including:

a. the domain $x > 0$

b. the x- and y-intercepts $\ln(x) = 0 \Rightarrow x = 1$

c. intervals on which the function is increasing and decreasing inc $(0, e)$ dec (e, ∞)

d. local maxima and/or minima $(e, 1/e)$

e. horizontal and vertical asymptotes. vertical at $x = 0$, horizontal at $y = 0$

$$f'(x) = \frac{u'v - v'u}{v^2} \qquad u = \ln(x) \quad v = x \qquad u' = 1/x \quad v' = 1$$

$$= \frac{1/x \cdot x - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}$$

$$f'(x) = 0 \Leftrightarrow \ln(x) = 1 \Leftrightarrow x = e$$

$f'(x) > 0$ when $\ln(x) < 1 \Leftrightarrow x < e$

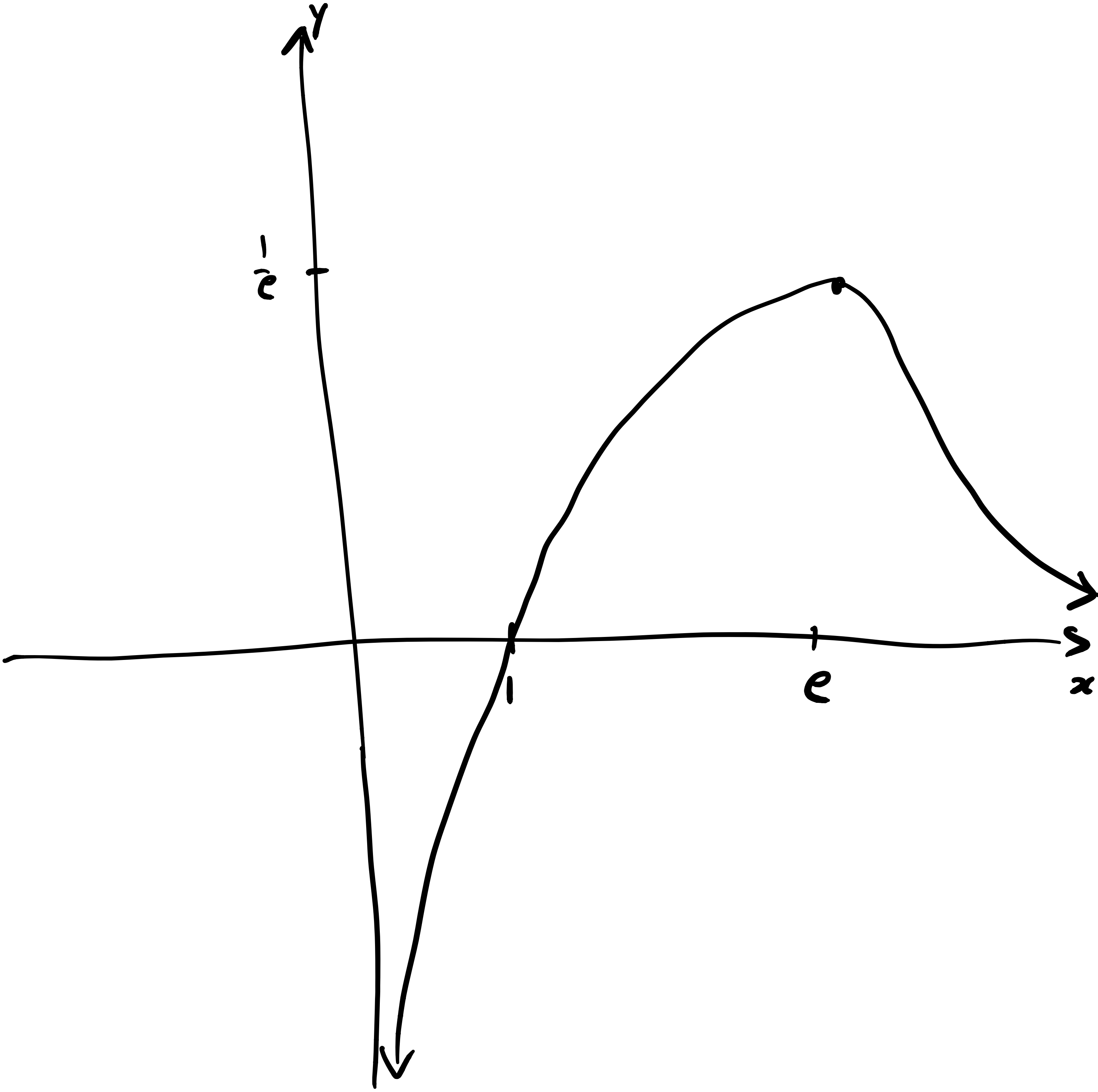
$f'(x) > 0$ when $\ln(x) > 1 \Leftrightarrow x > e$

So f is increasing on $(0, e)$ and decreasing on (e, ∞)

$\therefore (e, \frac{1}{e})$ is a max

Know $\frac{\ln(x)}{x} > 0$ for $x > 1$, but f decreasing on $(e, \infty) \Rightarrow$ horizontal asymptote (as $x \to \infty$, $f(x) \to 0$)

Know $\ln(x) \to -\infty$ as $x \to 0$, and $\frac{1}{x} \to \infty$ as $x \to 0$. So $\frac{\ln(x)}{x} \to -\infty$ as $x \to 0$



Question 2 (8 marks)

The number of rabbits increases according to the model $n(t) = Ae^{bt}$, where t is time in years, $n(t)$ is the population size at time t, A is the initial size of the population and b is the relative rate of growth.

Rabbits were introduced to a small island eight years ago. The current rabbit population on the island is estimated to be 4200, with a relative growth rate of 55% per year. $\rightarrow b = 0.55$

a. What was the initial size of the rabbit population?

$4200 = Ae^{0.55 \times 8} = Ae^{4.4}$

$\Rightarrow A = 4200e^{-4.4} \approx 52$ rabbits

b. Estimate the population 12 years from now. $8 + 12 = 20$

$n(20) = 52e^{0.55 \times 20}$

$\approx 3 \times 10^6$ rabbits

c. Determine when the population is increasing at a rate of 250 000 rabbits per year.

$n'(t) = Abe^{bt}$

$= 52 \times 0.55 \times e^{0.55t}$

$= 28.6\, e^{0.55t}$

$250\,000 = 28.6\, e^{0.55t}$

$\Rightarrow e^{0.55t} = \frac{250\,000}{28.6}$

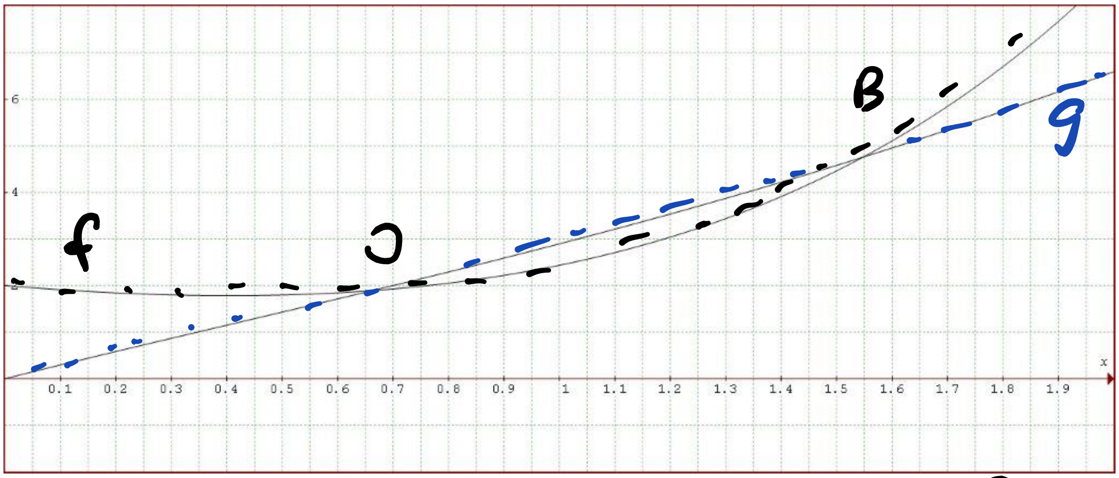
$\Rightarrow 0.55t = \ln\left(\frac{250\,000}{28.6}\right)$

$\Rightarrow t = \frac{1}{0.55} \ln\left(\frac{250\,000}{28.6}\right) \approx 16$ yrs

Question 3 (4 marks)

In the figure below, $f(x)$ and $g(x)$ intersect at O and B.

If $f(x) = 2e^{x} - 3x$ and $g(x) = -3e^{-x} + x^{2} + 3$, find the area of the region bounded by $f(x)$ and $g(x)$.



In the region we have $g(x) \geq f(x)$

So Area $= \int_{O}^{B} g(x) - f(x)\, dx$

Find coords of O & B:

$2e^{x} - 3x = -3e^{-x} + x^{2} + 3$

$\Rightarrow 2e^{x} + 3e^{-x} - 3x - x^{2} - 3 = 0$

Solve numerically: $x \simeq 0.66,\ 1.55$

So $O = (0.66,\ 1.89)$

$B = (1.55,\ 4.77)$

$\therefore$ Area $=$

$$\int_{0.66}^{1.55} -3e^{-x} + x^{2} + 3 - 2e^{x} + 3x \quad dx$$

$$= \left[3e^{-x} + \frac{x^3}{3} + 3x - 2e^{x} + \frac{3x^2}{2} \right]_{0.66}^{1.55}$$

$\vdots$

Calculator

$= 0.298649$ Units^2

Question 4 (9 marks)

The velocity of a particle (metres per second) is given by $v(t) = \frac{e^{2t}}{2} - 5t - \frac{1}{2}$

Determine:

a. the velocity of the particle after two seconds.

$$v(2) = \frac{e^4}{2} - 10 - \frac{1}{2}$$

$$= \frac{e^4}{2} - 10.5 \quad m/s$$

b. the acceleration of the particle after two seconds.

$$a(t) = v'(t) = \frac{2e^{2t}}{2} - 5 = e^{2t} - 5$$

$$\therefore a(2) = e^4 - 5 \quad m/s^2$$

c. when the acceleration is positive.

$$a(t) > 0 \iff e^{2t} - 5 > 0$$

$$\iff e^{2t} > 5$$

$$\iff 2t > \ln(5)$$

$$\iff t > \frac{\ln(5)}{2}$$

Question 5 (5 marks)

Calculate the value/s of x for which $f(x) = \ln(3x - 2) + 1$ and $g(x) = -4\cos(0.5x) + 2$, for $1 \leq x \leq 8$ have the same gradients.

$$f'(x) = \frac{3}{3x-2}$$

$$g'(x) = -4 \times 0.5 \times (-\sin(0.5x))$$

$$= 2\sin(0.5x)$$

Set $\frac{3}{3x-2} = 2\sin(0.5x)$

and solve numerically:

$x \approx 1.4395...$

$x \approx 6.0988...$

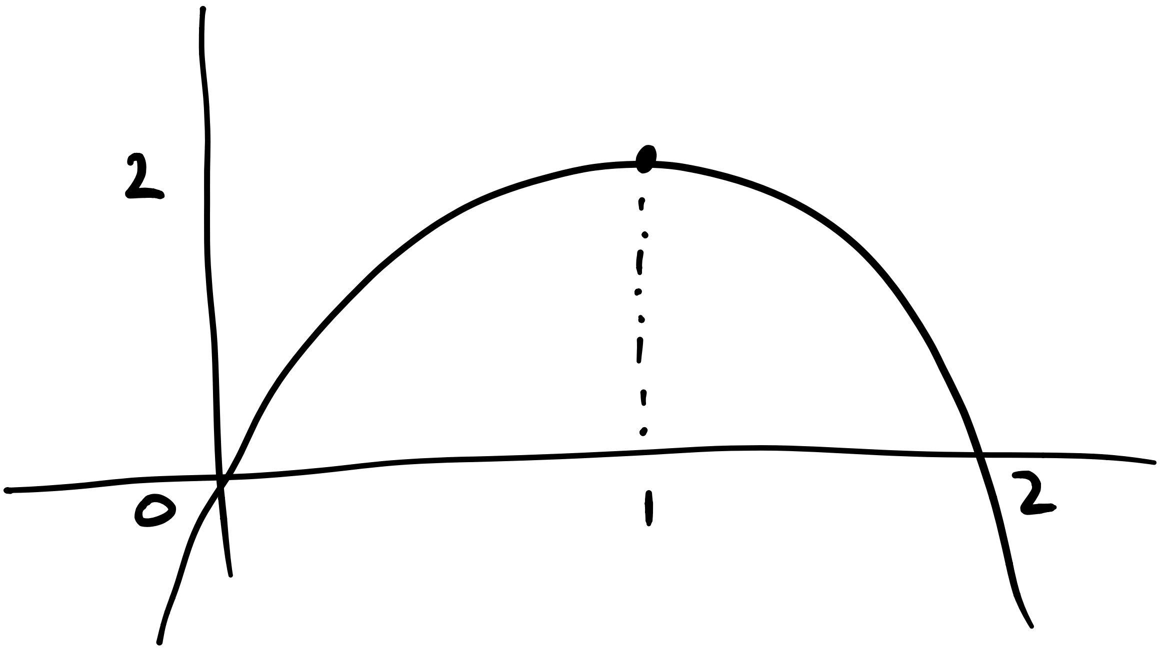
(keep within range $1 \leq x \leq 8$)

Question 6 (8 marks)

Determine the line $y{=}mx$ that divides the area under the curve $y = 2x(2-x)$ over $[0, 2]$ into two regions of equal area.

Justify all procedures and decisions by explaining mathematical reasoning.

Evaluate the reasonableness of your solution.



Area Under curve $= \int_0^2 4x - 2x^2 \, dx$

$= \frac{4x^2}{2} - \frac{2x^3}{3} \Big|_0^2$

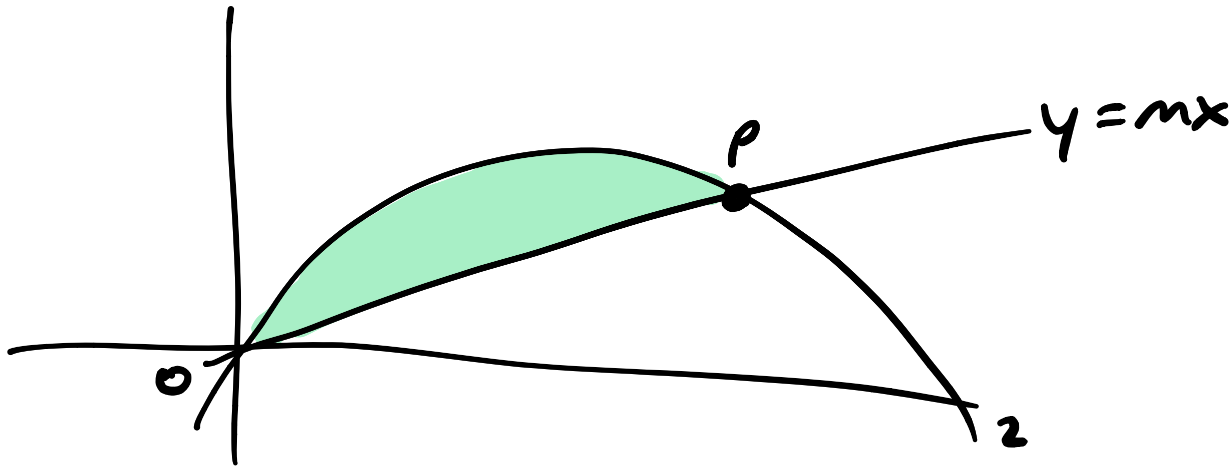
$= 2x^2 - \frac{2x^3}{3}$

$= 2 \times 4 - \frac{2 \times 8}{3}$

$= 8 - \frac{16}{3}$

$= 8/3$ units^2

We're looking to divide this region into two regions, each with area $4/3$ units^2



Suppose P, the point of intersection of line & parabola, has coords (k, mk)

Then $mk = 2k(2-k)$

$$\Rightarrow \quad m = \frac{2k(2-k)}{k} = 2(2-k) = 4-2k$$

and

$$\text{Shaded Area} = \int_0^k (4x-2x^2) - (4-2k)x \, dx$$

$$= \int_0^k 4x - 2x^2 - 4x + 2kx \, dx$$

$$= \int_0^k -2x^2 + 2kx \, dx$$

$$\therefore \frac{4}{3} = \left[-\frac{2}{3}x^3 + kx^2 \right]_0^k$$

$$= -\frac{2}{3}k^3 + k^3$$

$$= \frac{k^3}{3}$$

$\Rightarrow k^3 = 4$ and so $k = \sqrt[3]{4}$

Hence the equation of the line is

$$y = (4 - 2\sqrt[3]{4})\,x$$

Check reasonable: use technology to determine relevant areas

Question 7 (4 marks)

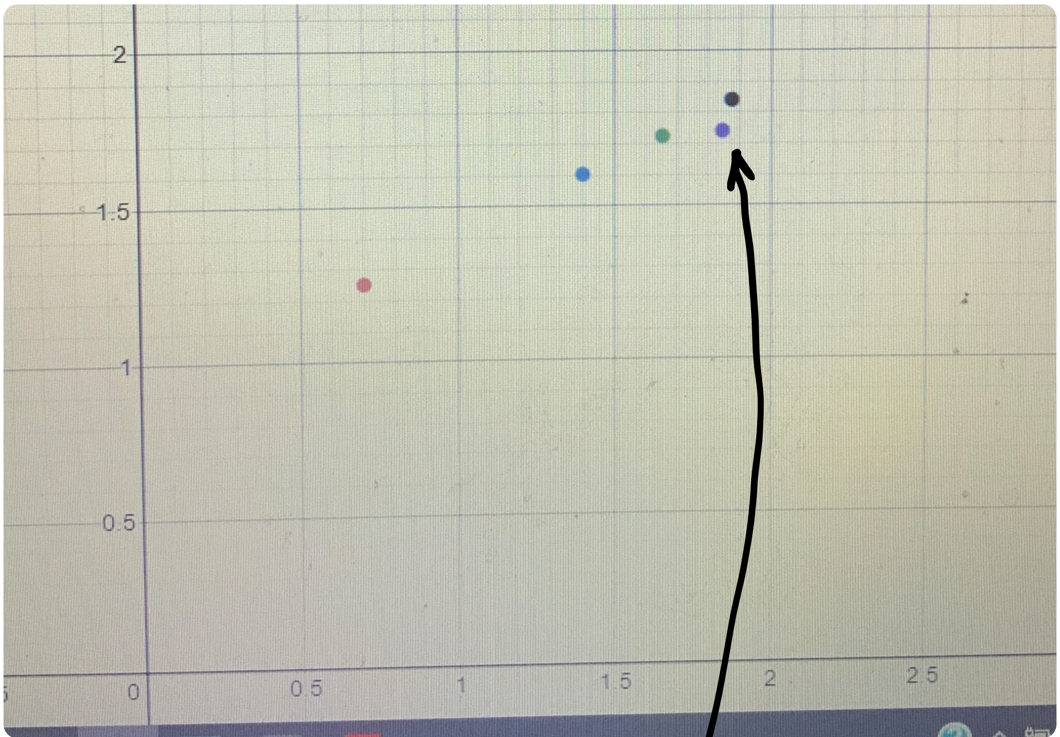
The velocity $V m/s$ of an object varies as the height h in metres from which the object is dropped as shown in the following table:

h (m)	5	25	45	70	75
V (m/s)	17.9	40	53.6	55	69.3

One of the given values of V is incorrect. If $\log_{10} V$ is a linear function of $\log_{10} h$, use a graphical method to locate the incorrect value and determine a more appropriate value of V.

$\log(h)$	0.699	1.398	1.653	1.845	1.875
$\log(V)$	1.253	1.602	1.724	1.740	1.840

Plot the points using graphics calculator, look for a straight line:



Problem!

The value for $h = 70$ produces the greatest residual for the linear regression model.

To determine a more appropriate value, use other values for linear regression to generate equation

$$\log(V) = 0.4975 \log(h) + 0.9051$$

Now sub $h = 70$ and solve for V:

$V = 66.5$

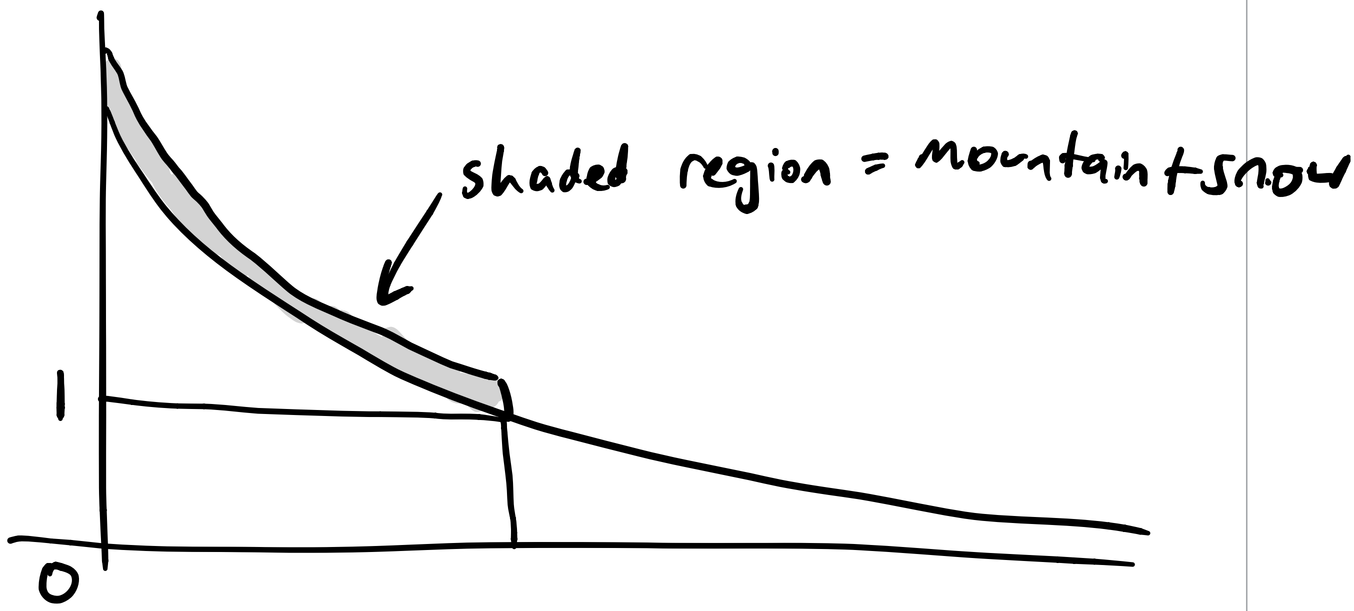
(exact value obtained depends on accuracy of data stored in calc)

Question 8 (7 marks)

Terry is an avid skier. The equation that best models his favourite run is given by $H = 1.8e^{-x} + 0.43$ where H is the height in kilometres above sea level and x represents the cross-sectional width of the run in kilometres.

The run terminates at a place that is 1 km above sea level. The cover of snow on the mountain is 2 metres (assume this cover is constant across the entire run). If the run is 300 m wide, calculate the volume of snow on the run.

Graph (use calc)



First, we Find the x-value when $y=1$ (at end of run)

So $1.8e^{-x} + 0.43 = 1$

$\Rightarrow x \approx 1.1499...$

Now, the curve's height is given in km, so the added height From snow results in an increase of 0.002

$\therefore$ equation of the new curve is

$$y = 1.8e^{-x} + 0.43 + 0.002$$

$$= 1.8e^{-x} + 0.432$$

The cross-sectional area is

end of run

$$\int_0^{1.1499} (y - H)\, dx$$

start of run

difference between run with & without snow

$$= \int_0^{1.1499} 0.002\, dx$$

$$= 0.002x \Big|_0^{1.1499}$$

$$= 0.0023 \text{ km}^2$$

So the volume of the snow is the cross sectional area multiplied by the width of the run (300m = 0.3 km)

$$\therefore V = 0.0023 \times 0.3 = 0.00069 \text{ km}^3$$

Examination marks summary

Paper 1 (technology-free)	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)
1a. b. c.	7		
1d.		4	
2	16		
3	11		
4	7		
5		6	
6		6	
7		5	
8			7
Totals	**41**	**21**	**7**

Paper 2 (technology-active)	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)
1	11		
2	8		
3	4		
4	9		
5		5	
6			8
7			4
8			7
Totals	**32**	**5**	**19**

Instrument-specific marking guide (ISMG)

Criterion: Foundational knowledge and problem-solving

Assessment objectives

1. select, recall and use facts, rules, definitions and procedures drawn from all Unit 3 topics
2. comprehend mathematical concepts and techniques drawn from all Unit 3 topics
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from all Unit 3 topics

The student work has the following characteristics:	Cut-off	Marks
• consistently correct selection, recall and use of facts, rules, definitions and procedures; authoritative and accurate command of mathematical concepts and techniques; astute evaluation of the reasonableness of solutions and use of mathematical reasoning to correctly justify procedures and decisions; and fluent application of mathematical concepts and techniques to solve problems in a comprehensive range of simple familiar, complex familiar and complex unfamiliar situations.	> 93%	15
	> 87%	14
• correct selection, recall and use of facts, rules, definitions and procedures; comprehension and clear communication of mathematical concepts and techniques; considered evaluation of the reasonableness of solutions and use of mathematical reasoning to justify procedures and decisions; and proficient application of mathematical concepts and techniques to solve problems in simple familiar, complex familiar and complex unfamiliar situations.	> 80%	13
	> 73%	12
• thorough selection, recall and use of facts, rules, definitions and procedures; comprehension and communication of mathematical concepts and techniques; evaluation of the reasonableness of solutions and use of mathematical reasoning to justify procedures and decisions; and application of mathematical concepts and techniques to solve problems in simple familiar and complex familiar situations.	> 67%	11
	> 60%	10
• selection, recall and use of facts, rules, definitions and procedures; comprehension and communication of mathematical concepts and techniques; evaluation of the reasonableness of some solutions using mathematical reasoning; and application of mathematical concepts and techniques to solve problems in simple familiar situations.	> 53%	9
	> 47%	8
• some selection, recall and use of facts, rules, definitions and procedures; basic comprehension and communication of mathematical concepts and techniques; inconsistent evaluation of the reasonableness of solutions using mathematical reasoning; and inconsistent application of mathematical concepts and techniques.	> 40%	7
	> 33%	6
• infrequent selection, recall and use of facts, rules, definitions and	> 27%	5

procedures; basic comprehension and communication of some mathematical concepts and techniques; some description of the reasonableness of results; and infrequent application of mathematical concepts and techniques.	> 20%	4
• isolated selection, recall and use of facts, rules, definitions and procedures; partial comprehension and communication of rudimentary mathematical concepts and techniques; superficial description of the reasonableness of results; and disjointed application of mathematical concepts and techniques.	> 13%	3
	> 7%	2
• isolated and inaccurate selection, recall and use of facts, rules, definitions and procedures; disjointed and unclear communication of mathematical concepts and techniques; and illogical description of the reasonableness of solutions.	> 0%	1
• does not satisfy any of the descriptors above.		0